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ON LIOUVILLE'S THEOREM FOR LOCALLY QUASIREGULAR MAPPINGS IN R^n

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ABSTRACT

Let $f : R^n \to R^n$ be locally quasiregular in the sense that the restriction of f to any ball $|x| < r$ has finite inner dilatation $K_I(r)$. Suppose that the growth condition $\int^\infty r^{-1} K_I(r)^{1/(1-n)} dr = \infty$ holds. Then Liouville's theorem is valid: *If f is bounded, f is a constant.* An example shows that this growth condition is relatively sharp.

1. Introduction

The classical theorem of Liouville is valid for quasiregular mappings (cf. [6, §3]):

If the function $f: R^n \to R^n$ is quasiregular and bounded in R^n, then f is constant.

Actually, a quasiregular mapping $f: R^n \to R^n$ is constant, if

$$\lim_{x \to \infty} |f(x)|/|x|^{K_I(f)^{1/(1-n)}} = 0 \tag{1}$$

where $1 \leqq K_I(f) < +\infty$ is the inner dilatation of f [5, theorem 3.7].

In this paper we are to consider the following situation. Suppose that the mapping $f: R^n \to R^n$ is locally quasiregular in R^n, i.e. that the restriction of f to every ball $B_r = \{x \in R^n \mid |x| < r\}$, $r > 0$, is quasiregular with the inner dilatation $1 \leqq K_I(r) < +\infty$. We shall prove that the growth condition

$$\int^\infty \frac{dr}{rK_I(r)^{1/(n-1)}} < +\infty \tag{2}$$

is necessary in order that the validity of the conclusion in Liouville's theorem should be violated.

The growth condition (2) is relatively exact; in [7] Zorič constructed an example showing that, if φ: $[0,\infty) \to [1,\infty)$ is a nondecreasing function with

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$$\int^{\infty} \frac{dr}{r\varphi(r)^{1/(n-1)}} < +\infty,$$

then there is a locally quasiconformal function $f: R^n \to \{y \in R^n \mid |y| < 1\}$ with the inner dilatation $K_I(r) \leqq \mathrm{const} \cdot \varphi(r)$ for large r. The constructed function is radial.

For the definition and properties of quasiregular mappings we refer the reader to [4]. We mainly use standard notation.

2. Liouville's theorem

Suppose that the mapping

$$f = (f_1, \ldots, f_n): R^n \to R^n \tag{3}$$

is locally quasiregular: (1) $f_1, \ldots, f_n \in C(R^n) \cap W^1_{n,\mathrm{loc}}(R^n)$ and (2) every restriction $f \mid \bar{B}_r$ has the outer dialatation $K_0(r) < +\infty$ and the inner dilatation $K_I(r) < +\infty$.

Thus we have the inequalities

$$|f'(x)|^n \leqq K_0(r) J(x,f) \qquad (|x| \leqq r) \tag{4}$$

and

$$J(x,f) \leqq K_I(r) l(f'(x))^n \qquad (|x| \leqq r) \tag{5}$$

for every $r > 0$ and for a.e. $x \in \bar{B}_r$. Here $f'(x)$ is the derivative of f, $J(x,f)$ the Jacobian determinant of f, and

$$|f'(x)| = \max_{|h|=1} |f'(x)h|, \qquad l(f'(x)) = \min_{|h|=1} |f'(x)h|$$

are the usual linear norms. The relations

$$K_0(r) \leqq K_I(r)^{n-1}, \qquad K_I(r) \leqq K_0(r)^{n-1} \tag{6}$$

are valid for all $r > 0$; cf. [4, (2.25)]. We also write $K(r) = \max\{K_0(r), K_I(r)\}$.

Now we state Liouville's theorem in the form:

THEOREM. (Liouville) *Suppose that $f: R^n \to R^n$ is locally quasiregular in R^n. If some coordinate function of f is bounded from below and if*

$$\int^{\infty} \frac{dr}{rK_I(r)^{1/(n-1)}} = +\infty, \tag{7}$$

then f is constant.

REMARK. Of course, the condition $\int^{\infty} r^{-1} K(r)^{1/(1-n)} dr = +\infty$ implies (7).

3. Proof of the theorem

Our proof of Liouville's theorem is based on the calculus of variations. According to Reshetnyak [6, (2.10)] each coordinate function $f_1, \ldots, f_n$ is a free extremal of the variational integral

$$I(u, D) = \int_D (\theta(x)\nabla u(x), \nabla u(x))^{n/2} dx \qquad (\bar{D} \subset R^n) \tag{8}$$

where the integrand is given by

$$(\theta(x)w, w)^{n/2} = \begin{cases} |f'(x)^{-1} w|_{R^n}^n J(x, f) & \text{when } J(x, f) \neq 0, \\ |w|^n, & \text{otherwise,} \end{cases} \tag{9}$$

for $w \in R^n$. Obviously, the mapping $w \to (\theta(x)w, w)^{n/2}$ is convex for a.e. fixed $x \in R^n$. By (4) and (5) we have for a.e. x

$$\frac{|w|^n}{K_0(|x|)} \leqq (\theta(x)w, w)^{n/2} \leqq K_I(|x|)|w|^n, \tag{10}$$

when $w \in R^n$.

Let u denote a coordinate function of f that is bounded from below. If u is constant, so is f by (4). Without loss of generality we may assume that $u \geqq \sqrt[n]{n-1}$. Since u is monotone, so is $\log u$. By (10)

$$\frac{|\nabla \log u(x)|^n}{K_0(|x|)} \leqq \frac{(\theta(x)\nabla u(x), \nabla u(x))^{n/2}}{(u(x))^n},$$

thus [3, lemma 8], a simple consequence of Gehring's and Mostow's oscillation inequality, yields

$$\operatorname{osc}^n (\log u, B_r) \int_r^R \frac{d\rho}{\rho K_0(\rho)} \leqq A \int_{r<|x|<R} \frac{(\theta(x)\nabla u(x), \nabla u(x))^{n/2}}{(u(x))^n} dx \tag{11}$$

for all $0 < r < R$; the constant A depends only on the dimension n.

In order to estimate the right-hand member of (11) we choose a test-function $\zeta \in C(R^n) \cap W_n^1(R^n)$, $0 \leqq \zeta \leqq 1$, such that $\zeta(x) = 1$, when $|x| \leqq R$. Let $L > R$ and require that $\zeta(x) = 0$, when $|x| \geqq L$. The function

$$v = u + \zeta^n u^{-(n-1)}$$

has the generalized derivative

$$\nabla v = \left(1-(n-1)\frac{\zeta^n}{u^n}\right)\nabla u + n\frac{\zeta^{n-1}}{u^{n-1}}\nabla\zeta.$$

Since $u = v$ in $\{x \mid |x| \geqq L\}$ and u is a free extremal of (8), the inequality

$$I(u,\bar{B}_L) \leqq I(v,\bar{B}_L) \tag{12}$$

is valid. By the convexity of the integrand (9) and by (10), we have

$$(\theta\nabla v,\nabla v)^{n/2} \leqq \left(1-(n-1)\frac{\zeta^n}{u^n}\right)(\theta\nabla u,\nabla u)^{n/2} + \frac{n^n}{(n-1)^{n-1}}K_I|\nabla\zeta|^n. \tag{13}$$

Integrating (13) over $\bar{B}_L$ and using (12) we get the bound

$$\int_{B_R}\frac{(\theta\nabla u,\nabla u)^{n/2}}{u^n}dm \leqq \left(\frac{n}{n-1}\right)^n\int_{R<|x|<L}K_I(|x|)|\nabla\zeta(x)|^n dx. \tag{14}$$

Indeed,

$$\inf_{\zeta}\int K_I|\nabla\zeta|^n dm \leqq \omega_{n-1}\left(\int_R^L\frac{d\rho}{\rho K_I(\rho)^{1/(n-1)}}\right)^{1-n} \tag{15}$$

where the infimum is taken over all admissible ζ and ω_{n-1} is the area of the unit sphere. The infimum in (15) can be regarded as a weighted capacity of the condenser $(B_L, \bar{B}_R)$. (It is easily seen that equality, actually, holds in (15).) In fact, we only need to consider the function

$$1-\zeta(x) = \frac{\displaystyle\int_R^{|x|}\frac{d\rho}{\rho K_I(\rho)^{1/(n-1)}}}{\displaystyle\int_R^L\frac{d\rho}{\rho K_I(\rho)^{1/(n-1)}}}, \qquad R\leqq|x|\leqq L,$$

for proving (15).

Combining (14), (15), and (11) we get

$$\operatorname{osc}^n(\log u, B_r) \leqq \frac{2\omega_{n-1}eA}{\displaystyle\int_r^R\frac{d\rho}{\rho K_0(\rho)}\left(\int_R^L\frac{d\rho}{\rho K_I(\rho)^{1/(n-1)}}\right)^{n-1}} \tag{16}$$

for all $0<r<R<L$. Let $L\to+\infty$. If (7) is valid, then we must have $\operatorname{osc}^n(\log u, B_r)=0$, and so $\log u$, and thus also u, is constant in B_r. Since $r>0$ was arbitrary, this is the desired result. (As

$$\int^{\infty}\frac{d\rho}{\rho K_I(\rho)^{1/(n-1)}} \geqq \int^{\infty}\frac{d\rho}{\rho K_0(\rho)} \tag{17}$$

by (6), no further essential information can be extracted from (16).)

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